

# Experience in Factoring Large Integers Using Quadratic Sieve

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## Abstract

GQS is a computer program for factoring “large” integers. It is based on multiple polynomial quadratic sieve. The current version, 2.0, can factor a 82-decimal-digit integer in a PC with AMD 1.8G Hz processor and 512 MB main memory in one day. The largest number I have factored using GQS is an 115-digit integer. This was done in a PC cluster of about 50 nodes. In this talk I will describe how the GQS is implemented and my experience in using GQS.

# Theory of Quadratic Sieve

Quadratic sieve is an “efficient” factoring algorithm for integers up to about 110 decimal digits.

The algorithm first finds a pair of congruent squares

$$x^2 \equiv y^2 \pmod{n},$$

where  $n$  is the integer to be factored.

Suppose that

$$x \not\equiv \pm y \pmod{n}$$

Then

$$n = \gcd(n, x - y) \times \gcd(n, x + y).$$

## An Example

$$n = 33221, B = \{2, 3, 5, 7\}, \quad x_i = \left[ \sqrt{33221} \right] + i = 182 + i.$$

$$\begin{array}{lll} (189)^2 & \equiv 2500 & \equiv (2)^2(5)^4 \\ (378)^2 & \equiv 10000 & \equiv (2)^4(5)^4 \\ (409)^2 & \equiv 1176 & \equiv (2)^3(3)^1(7)^2 \\ (567)^2 & \equiv 22500 & \equiv (2)^2(3)^2(5)^4 \\ (682)^2 & \equiv 30 & \equiv (2)^1(3)^1(5)^1 \\ (802)^2 & \equiv 12005 & \equiv (5)^1(7)^4 \\ (818)^2 & \equiv 4704 & \equiv (2)^5(3)^1(7)^2 \\ (835)^2 & \equiv 32805 & \equiv (3)^8(5)^1 \\ (845)^2 & \equiv 16384 & \equiv (2)^{14} \\ (983)^2 & \equiv 2880 & \equiv (2)^6(3)^2(5)^1 \\ (1169)^2 & \equiv 4500 & \equiv (2)^2(3)^2(5)^3 \\ & \vdots & \end{array}$$

## An Example: Factoring

$$189^2 \equiv 2^2 5^4$$

$$189^2 \equiv 2^2 5^4 \equiv 50^2$$

$$189^2 - 50^2 \equiv 0 \pmod{33221}$$

$$(189 - 50)(189 + 50) \equiv 0 \pmod{33221}$$

$$(139)(239) = k(33221)$$

$$33221 = 139 \times 239$$

$$802^2 \equiv 5^1 7^4$$

$$835^2 \equiv 3^8 5^1$$

$$(802 \cdot 835)^2 \equiv (3^4 \cdot 5 \cdot 7^2)^2$$

$$(669670)^2 \equiv (19845)^2$$

$$(5250)^2 \equiv (19845)^2$$

$$(5250 - 19845)(5250 + 19845) = k \cdot 33221$$

$$(-14595)(20595) = k \cdot 33221$$

$$(-14595, 33221) = 139$$

$$(20595, 33221) = 239$$

$$409^2 \equiv 2^3 3^1 7^2$$

$$682^2 \equiv 2^1 3^1 5^1$$

$$835^2 \equiv 3^8 5^1$$

$$(409 \cdot 682 \cdot 835)^2 \equiv (2^2 \cdot 3^5 \cdot 5 \cdot 7)^2$$

$$(232913230)^2 \equiv (34020)^2$$

$$(799)^2 \equiv (799)^2$$

## Theory of Quadratic Sieve

Assume that  $n$  is a product of two different primes.

$$x^2 \equiv y^2 \pmod{n} \Rightarrow x^2 - y^2 \equiv 0 \pmod{n} \Rightarrow n \mid (x - y)(x + y)$$

$$x \not\equiv \pm y \pmod{n} \Rightarrow n \nmid (x - y) \quad \text{and} \quad n \nmid (x + y)$$

Thus,  $x - y$  and  $x + y$  each contains only one factor of  $n$ , but not both.

Therefore,  $\gcd(n, x - y)$  and  $\gcd(n, x + y)$  are non-trivial factors of  $n$ .

## Find Congruent Squares

First, find a set of integers whose square modulo  $n$  are  $b$ -smooth:

$$S = \{x \mid \text{every prime factor of } x^2 \bmod n \text{ is bounded by } b\}.$$

Let  $P = \{p \mid p \text{ is prime and } p \leq b\}$ .

The residue of the square of each  $x_i \in S$  can be factored by using the primes in  $P$ .

$$x_i^2 \bmod n = \prod_{j=1}^s p_j^{e_{i,j}}.$$

The next step is to find a subset  $T$  of  $S$  such that

$$\prod_{x_i \in T} x_i^2 \pmod n = \prod_{x_i \in T} \left( \prod_{j=1}^s p_j^{e_{i,j}} \right) = \prod_{j=1}^s p_j^{c_j},$$

in which each  $c_j = \sum_{x_i \in T} e_{i,j}$  is even.

After  $T$  is computed, we can then find two congruent squares

$$\left( \prod_{x_i \in T} x_i \right)^2 \equiv \left( \prod_{j=1}^s p_j^{c_j/2} \right)^2 \pmod n.$$

If  $|S| > |P|$ , then there exists a subset  $T \subset S$  satisfying the the above conditions.

## Find $b$ -Smooth Integers — Simple Method

**for**  $i = 1, 2, \dots$  **do**

$$x_i = \lfloor \sqrt{n} \rfloor + i$$

$$r = x_i^2 \bmod n$$

**for**  $j = 1, 2, \dots, s$  **do**

$$e_{i,j} = 0$$

**while**  $(p_j \mid r)$   $\{e_{i,j} = e_{i,j} + 1; r = r/p_j;\}$

**if**  $(r = 1)$  **then**  $x_i^2 \bmod n = \prod_{j=1}^s p_j^{e_{i,j}}$

It takes *one week* to factor a 43-digit integer in a PC.

## Implementation of GQS 2.0

1. The program computes a *multiplier*  $k$ , and try to factor  $kn$ , instead of  $n$ . The speed of the computation can be 3 times faster by selecting a proper value of  $k$ .
2. The factor basis  $P$  is chosen as the set of the first  $s$  primes  $p$  such that  $n$  is a quadratic residue modulo  $p$ .
3. The set  $S$  is computed by sieving with *multiple polynomials*.
4. The set  $T$  is computed by Gaussian elimination over  $\mathbf{Z}/2\mathbf{Z}$ .
5. In addition to the *fully factored* relations ( $r = 1$ ), relations with 1, 2, or even 3 large primes ( $1 < r < p_m$ ), are also collected, where  $p_m$  is the upper bound for the primes. These are called *partially factored* relations.

## Six Programs in GQS

**gqsinit** determines the size  $s$  of the factor basis  $P$ , the value of the multiplier  $k$ , and the length  $m$  of the sieve interval.

It can also generate a number  $n$ , which is a product of two random primes, to be factored for testing the factoring programs.

**gqssieve** computes a set of numbers  $S$  whose square modulo  $kn$  is “ $p_s$ -smooth”.

It is the program that runs most of the time.

The program *gqssieve* is designed to be able to continue its computation from the last time it was interrupted.

It can also be run in *parallel* in one machine with multiple CPU's or in different machines, such as a PC cluster.

**gqsmerge** combines all the relations computed by *gqssieve*. It also merges all the large primes in partially factored relations into the factor basis.

**gqscycle** eliminates useless relations.

A relation is *useless* if it has a prime  $p$  with odd power but no other relations have it.

**gqsreduce** reduces the size of the factor basis.

This is an effective way to reduce the size of the system of linear equations need to be solved in the factoring.

**gqsfactor** computes the subset of relations  $T$ , and then computes two congruent squares  $x$  and  $y$ , and finally compute the factors of  $n$ .

## Implementation of gqsieve

GQS SIEVE 2.0 uses multiple pair of polynomials  $(f(x), g(x))$  for sieving.

Each pair of polynomials  $f(x)$  and  $g(x)$  must satisfy the following conditions.

1.  $f(x)$  is a square,
2.  $f(x) \neq g(x)$  but  $f(x) \equiv g(x) \pmod{n}$ .
3. The value of  $g(x)$  should be small in the sieve interval  $[-m, m]$ .

In the simple method given above,

$$\begin{aligned}f(x) &= (x + \lfloor \sqrt{n} \rfloor)^2, \\g(x) &= x^2 + 2\lfloor \sqrt{x} \rfloor + (\lfloor \sqrt{n} \rfloor)^2 - kn.\end{aligned}$$

The problem of using one pair of polynomials is that when  $x$  gets large,  $g(x)$  will also be large. Large integers are unlikely to be smooth.

## Generation of the Polynomials

Let

$$\begin{aligned}h(x) &= (ax + b)^2 = a^2x^2 + 2abx + b^2, \\g(x) &= ax^2 + 2bx + c, \\a \cdot g(x) &= a^2x^2 + 2abx + ac.\end{aligned}$$

If  $b^2 \equiv ac \pmod{n}$  then  $h(x) \equiv a \cdot g(x) \pmod{n}$ .

Assume that  $a$  is a square,  $a \equiv d^2 \pmod{n}$ .

Let  $f(x) = (ax + b)^2(d^{-1})^2 = (ad^{-1}x + bd^{-1})^2$ .

Then  $f(x) \equiv g(x) \pmod{n}$ .

## Generation of “good” Polynomials

1. minimize  $\sup |g(x)|$  over  $[-m, m]$ , or

2. minimize  $\int_{-m}^m |g(x)| dx$ , or

3. minimize  $\int_{-m}^m g^2(x) dx$ ,

subject to  $b^2 - ac = kn$  (Note that  $b^2 \equiv ac \pmod{n}$ .)

The *optimal* solution is

$$a = \frac{\sqrt{2}}{2} \sqrt{kn}/m, \quad b = 0, \quad c = -\frac{\sqrt{2}}{4} m \sqrt{kn}.$$

Compute  $d \approx \sqrt{\sqrt{kn/2/m}}$ , such that  $d$  is prime,  $\left(\frac{kn}{d}\right) = 1$ .

Let  $a = d^2$ .

Find  $b$  such that  $b^2 \equiv kn \pmod{a}$ .

$$b^2 - kn \equiv 0 \pmod{a} \Rightarrow b^2 - kn = ca \Rightarrow c = \frac{b^2 - kn}{a}.$$

$$g(x) = ax^2 + 2bx + c$$

Since  $b^2 - ac = kn$ ,  $f(x) \equiv g(x) \pmod{n}$ .

## Compute $b$

Compute  $h = (kn)^{(d+1)/4}$ , that is,  $h^2 = kn \pmod{d}$ .

Compute  $v = (2h)^{-1} \left( \frac{kn - h^2}{d} \right) \pmod{d}$ .

Compute  $b = h + vd \pmod{d^2}$ .

$$b^2 \equiv h^2 + 2hvd + v^2d^2 \equiv kn \pmod{d^2}$$

**Theorem.** Let  $p$  be a prime and  $f(x)$  be a polynomial with integer coefficients. If  $f(a) \equiv 0 \pmod{p^i}$  and  $f'(a) \not\equiv 0 \pmod{p}$ , then there exists a unique  $t$  so that  $f(a + tp^i) \equiv 0 \pmod{p^{i+1}}$ .

By the above theorem, we can “lift” a solution to a higher power by solving a linear congruent equation

$$f'(a)t \equiv -\frac{f(a)}{p^i} \pmod{p}.$$

Let  $r$  be a solution to  $x^2 \equiv w \pmod{p^i}$ , and  $f(x) = x^2 - w$ .

Need to solve a linear equation to lift the solution to  $x^2 \equiv w \pmod{p^{i+1}}$ .

$$2rt \equiv -\frac{r^2 - w}{p^i} \pmod{p}.$$

If  $(2r, p) = 1$ , then  $t \equiv -\frac{r^2 - w}{p^i} (2r)^{-1} \pmod{p}$ .

## Sieve

Problem: Find  $k$  such that  $g(k)$  is smooth.

If  $p^\alpha \mid g(x)$  then  $p^\alpha \mid g(x + jp^\alpha)$  for any integer  $j$ .

In GQS, sieve is done by

1. initializing an array  $w[-m..m]$  to 0, and
2. add  $\lfloor \alpha \log(p_j) + 0.5 \rfloor$  to  $w[x]$  iff  $p_j^\alpha \mid g(x)$ .

Ideally, we should do the sieve for all primes  $p \in P$  for all  $\alpha$  such that  $p^\alpha < b$ .

However, for  $\alpha > 1$ , GQS only do this for small set of  $p$  to reduce the computational time.

	2	3	5	7	4	9
$g(-m)$	v					
$g(-m + 1)$		v	v			v
$g(-m + 2)$	v				v	
$g(-m + 3)$				v		
$g(-m + 4)$	v	v				
$g(-m + 5)$						
$g(-m + 6)$	v		v		v	
$g(-m + 7)$		v				
$g(-m + 8)$	v					
$g(-m + 9)$						
$g(-m + 10)$	v	v		v	v	v
$g(-m + 11)$			v			
$g(-m + 12)$	v					
$g(-m + 13)$		v				
$g(-m + 14)$	v				v	
$g(-m + 15)$						
$\vdots$						
$g(m)$	v	v	v			

## Trial Division

Trial division is applied to test the smoothness of each  $g(k)$  when

$$x[k] > \left\lfloor \log \frac{m\sqrt{kn/2}}{p_s^t} \right\rfloor,$$

where  $t$  is the *threshold* value.

If  $t \leq 2$  then most relations will be fully factored.

If  $t > 2$ , then relations may not be fully factored, but the partial factorization may also be useful.

## Parallel Sieve

Sieve is very easy to be parallelized.

Let each process  $i$  do the sieve with the polynomial  $g(x) = ax^2 + bx + c$  in which  $a$  is within a given interval  $[l_i, r_i]$ .

Our experience shows that sieve is almost  $k$  times faster by using  $k$  computers.

## Check Point and Restart

It usually take a long time to do the sieve when the number to be factored is large.

Checkpoint and restart are very easy to implement.

Save the value of  $a$ , and the program can start from there.

## Implementation of *gqsfactor*

GQS FACTOR 2.0 use Gaussian elimination to find linear dependency after enough relations are found, and then factor  $n$ .

Each relation  $x_i$  in  $S$  is mapped to a vector  $v_i$  in  $(\mathbf{Z}/2\mathbf{Z})^s$ :

$$x_i = \prod_{j=1}^s p_j^{e_{i,j}} \quad \mapsto \quad v_i = (e_{i,1}, e_{i,2}, \dots, e_{i,s}).$$

$$n = 33221, B = \{2, 3, 5, 7\}$$

$(409)^2$	$\equiv 1176$	$\equiv (2)^3(3)^1(7)^2$	$(1, 1, 0, 0)$
$(567)^2$	$\equiv 22500$	$\equiv (2)^2(3)^2(5)^4$	$(0, 0, 0, 0)$
$(682)^2$	$\equiv 30$	$\equiv (2)^1(3)^1(5)^1$	$(1, 1, 1, 0)$
$(802)^2$	$\equiv 12005$	$\equiv (5)^1(7)^4$	$(0, 0, 1, 0)$
$(818)^2$	$\equiv 4704$	$\equiv (2)^5(3)^1(7)^2$	$(1, 1, 0, 0)$
$(835)^2$	$\equiv 32805$	$\equiv (3)^8(5)^1$	$(0, 0, 1, 0)$
$(845)^2$	$\equiv 16384$	$\equiv (2)^{14}$	$(0, 0, 0, 0)$
$(983)^2$	$\equiv 2880$	$\equiv (2)^6(3)^2(5)^1$	$(0, 0, 1, 0)$
$(1169)^2$	$\equiv 4500$	$\equiv (2)^2(3)^2(5)^3$	$(0, 0, 1, 0)$
$(1223)^2$	$\equiv 784$	$\equiv (2)^4(7)^2$	$(0, 0, 0, 0)$
$(1227)^2$	$\equiv 10584$	$\equiv (2)^3(3)^3(7)^2$	$(1, 1, 0, 0)$
$(1327)^2$	$\equiv 216$	$\equiv (2)^3(3)^3$	$(1, 1, 0, 0)$

Assume that more than  $|P| = s$  relations have been collected.  
These vectors cannot be linear independent.

Example of dependence:

$$\begin{array}{llll} (409)^2 \equiv 1176 & \equiv (2)^3(3)^1(7)^2 & (1, 1, 0, 0) \\ (682)^2 \equiv 30 & \equiv (2)^1(3)^1(5)^1 & (1, 1, 1, 0) \\ (835)^2 \equiv 32805 & \equiv (3)^8(5)^1 & (0, 0, 1, 0) \end{array}$$

Assume that there are  $t$  relations,  $t > s$ . A linear dependency can be found by solving the following equation for  $\delta_i$ .

$$\delta_1 v_1 + \delta_2 v_2 + \cdots + \delta_t v_t = 0.$$

This is equivalent to solving a system of linear equation

$$A\delta = 0.$$

where

$$A = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_t \end{pmatrix}^T.$$

After Gaussian elimination, every column after  $s$  yields a solution.

We can also let

$$A = \left( \begin{array}{c|c} v_1 & \\ v_2 & \\ \vdots & \\ v_t & \mathbf{I} \end{array} \right),$$

where  $\mathbf{I}$  is a  $t \times t$  identity matrix.

After Gaussian elimination, any zero-row in the first  $s$  elements yields a solution.

## Performance of GQS 2.0

I have factored an 82-digit integer in one day by a PC with AMD 1.8G Hz CPU and 512 MB memory.

I have also factored an 85-digit integer in 3 days in the same computer.

I have successfully factored a 110-digit integer. This was done in a PC cluster with about 50 available nodes. It took about a week to do the sieving. The merging of relations and the factoring took about a few hours.

The largest number I have factored is an 115-digit integer. This is done in the same PC cluster in about three weeks.

## Estimated Time for Factoring Integers Using GQS in a PC

$\log n$	50	60	70	80	90
time	0.01 h	0.16 h	2 h	22 h	210 h

$\log n$	100	110	120	130
time	80 d	600 d	4400 d	31000 d

Our experience shows that the estimation is quite close to the time required for factoring integers up to 110 digits.

## Another Way to Generate the Polynomials

Let  $n$  be the integer to be factored,  $m$  be the length of the sieve interval. Let

$$\begin{aligned} a &\approx \sqrt{n/2} / m, \\ b^2 &\equiv n \pmod{a}, \quad 0 < b < \frac{a}{2}, \\ c &= \frac{b^2 - kn}{a}. \end{aligned}$$

Let

$$f(x) = (ax + b)^2, \tag{1}$$

$$g(x) = ax^2 + 2bx + c. \tag{2}$$

$$f(x) = a^2x^2 + 2abx + b^2, \quad (3)$$

$$a \cdot g(x) = a^2x^2 + 2abx + ac. \quad (4)$$

The difference of the above two equations, (4) – (3), is

$$b^2 - ac = kn.$$

Therefore,

$$f(x) \equiv a \cdot g(x) \pmod{n}.$$

If  $a$  is a product of primes in the factor basis  $P$ , then we can sieve the polynomial  $g$  without inverting  $a$ .

1. Find a set of primes powers whose product is a “good” approximation to  $\sqrt{n/2}/m$ .
2. Solve  $f(x) = ax^2 + 2bx + c \equiv 0 \pmod{p^\alpha}$  efficiently.

The advantage is that many values of  $b$  can be found for one value of  $a$ , and the next value of  $b$  can be computed by a small amount of work from the previous one.

Our experience shows that for factoring integers up to 90 digits, the new method is about 20

For integers larger than 100 digits, the method is not much faster than the old method.

This is due to the time for finding a good approximation to  $\sqrt{n/2}/m$  become too large to be ignored.

# The Challenges for Factoring Large Integers

For very large  $n$ , the factor basis  $P$  will be very large in order to find relations efficiently.

$n$	50	60	70	80	90
$ P $	3000	4000	7000	15000	30000

$n$	100	110	120	130
$ P $	51000	120000	245000	525000

In addition to sieving, the space and the time required to solve the system of linear equations will be the obstacles for factoring large integers.

## Lanczos algorithm

Lanczos algorithm is a method for solving

$$Ax = b$$

where  $A$  is a symmetric  $n \times n$  matrix.

$$\begin{aligned}w_0 &= b, \\v_1 &= Aw_0, \\w_1 &= v_1 - \frac{(v_1, v_1)}{(w_0, v_1)}w_0,\end{aligned}$$

Then, for  $i > 0$ ,

$$\begin{aligned}v_{i+1} &= Aw_i, \\w_{i+1} &= v_{i+1} - \frac{(v_{i+1}, v_{i+1})}{(w_i, v_{i+1})}w_i - \frac{(v_{i+1}, v_i)}{(w_{i-1}, v_i)}w_{i-1},\end{aligned}$$

The algorithm stops when it finds a  $w_j$  such that

$$(w_j, Aw_j) = 0,$$

If  $w_j = 0$ , then

$$x = \sum_{i=0}^{j-1} \frac{(w_i, b)}{(w_i, v_{i+1})}w_i.$$

However, the matrix we are interested is generally not square and not symmetric.

Assume that we want to solve

$$Bx = y,$$

where  $B$  is an  $m \times n$  matrix,  $m \geq n$ .

We first embed the field  $\mathbf{Z}/2\mathbf{Z}$  to a larger field  $\mathbf{F}$  with more than  $n$  elements.

We then randomly choose an  $m \times m$  diagonal matrix  $D$  with nonzero elements in  $\mathbf{F}$ , and let

$$A = B^T D^2 B, \quad w = B^T D^2 y.$$

## Reduction of the Linear System

Let  $Q \subseteq P$  be a subset of primes in the factor basis.

Do the Gaussian elimination with respect to the primes in  $Q$  only.

This will not only reduce the space in the Gaussian elimination, it will also reduce the computational time.

$(409)^2$	$\equiv 1176$	$\equiv (2)^3(3)^1(7)^2$	$(1, 1, 0, 0)$
$(567)^2$	$\equiv 22500$	$\equiv (2)^2(3)^2(5)^4$	$(0, 0, 0, 0)$
$(682)^2$	$\equiv 30$	$\equiv (2)^1(3)^1(5)^1$	$(1, 1, 1, 0)$
$(802)^2$	$\equiv 12005$	$\equiv (5)^1(7)^4$	$(0, 0, 1, 0)$
$(818)^2$	$\equiv 4704$	$\equiv (2)^5(3)^1(7)^2$	$(1, 1, 0, 0)$
$(835)^2$	$\equiv 32805$	$\equiv (3)^8(5)^1$	$(0, 0, 1, 0)$
$(845)^2$	$\equiv 16384$	$\equiv (2)^{14}$	$(0, 0, 0, 0)$
$(983)^2$	$\equiv 2880$	$\equiv (2)^6(3)^2(5)^1$	$(0, 0, 1, 0)$
$(1169)^2$	$\equiv 4500$	$\equiv (2)^2(3)^2(5)^3$	$(0, 0, 1, 0)$
$(1223)^2$	$\equiv 784$	$\equiv (2)^4(7)^2$	$(0, 0, 0, 0)$
$(1227)^2$	$\equiv 10584$	$\equiv (2)^3(3)^3(7)^2$	$(1, 1, 0, 0)$
$(1327)^2$	$\equiv 216$	$\equiv (2)^3(3)^3$	$(1, 1, 0, 0)$
$(1364)^2$	$\equiv 120$	$\equiv (2)^3(3)^1(5)^1$	$(1, 1, 1, 0)$
$(1568)^2$	$\equiv 270$	$\equiv (2)^1(3)^3(5)^1$	$(1, 1, 1, 0)$
$(1589)^2$	$\equiv 125$	$\equiv (5)^3$	$(0, 0, 1, 0)$

In the factoring of the 115-digit integer, 192561 primes were used, and more than 123000000 relations had been collected. After elimination of useless relations, more than 480000 relations remain. The size of the final set of primes is about 450000.

The factorization is done by reducing 20000 primes at a time, until 80000 primes remain. Then gqsfactor was used to do the factoring.

## Acknowledgments

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## References

1. Henri Cohen. *A Course in Computational Algebraic Number Theory*. Springer-Verlag, 1993.
2. Hans Riesel. *Prime Numbers and Computer Methods for Factorization*. Birkhäuser, 1994.
3. Robert D. Silverman. The multiple polynomial quadratic sieve. *Mathematics of Computation*, 48(117):329–339, January 1987.